The Meaning of Market Efficiency

Robert Jarrow[*] Martin Larsson[†]

February 23, 2011

Abstract

Fama (1970) defined an efficient market as one in which prices always 'fully reflect' available information. This paper formalizes this definition and provides various characterizations relating to equilibrium models, profitable trading strategies, and equivalent martingale measures. These various characterizations facilitate new insights and theorems relating to efficient markets. In particular, in contrast to common belief, we show that one can test for an efficient market without the need to assume a particular equilibrium asset pricing model. Indeed, an efficient market is completely characterized by the absence of arbitrage opportunities and dominated securities. Other theorems useful for derivatives pricing are also provided.

KEY WORDS: efficient markets, information sets, strong-form efficiency, semi-strong form efficiency, weak-form efficiency, martingale measures, local martingale measures, no arbitrage, no dominance, economic equilibrium.

1 Introduction

The original definition of market efficiency is given by Fama [22], p. 383 in his seminal paper:

> "A market in which prices always 'fully reflect' available information is called 'efficient'."

Three information sets have been considered when discussing efficient markets[1]: (i) historical prices (weak form efficiency), (ii) publicly available information (semi-strong efficiency), and (iii) private information (strong form efficiency). A market may or may not be efficient with respect to each of these information sets.[2]

[*]Johnson Graduate School of Management, Cornell University, Ithaca, NY, 14853 and Kamakura Corporation

[†]School of Operations Research, Cornell University, Ithaca, NY, 14853

[1]This partitioning of the information sets is attributed to Harry Roberts, unpublished paper presented at the Seminar of the Analysis of Security Prices, U. of Chicago, May 1967 (see Fama (1970)).

[2]Market efficiency is closely related to the notion of a Rational Expectations Equilibrium (REE) where equilibrium prices reveal private information. A fully revealing REE is one where prices reveal all private information, analogous to a market that is strong-form efficient. A partially revealing REE is one where prices only partially reveal all private information, corresponding to semi-strong form efficiency (see Jordan and Radner [39] and Admati [1] for reviews). This relationship is discussed further in section 2 below.

To test market efficiency, it is commonly believed (see, for example, Campbell, Lo and MacKinlay [5] and Fama [24]) that one must first specify an equilibrium model, indeed Fama [23], p. 1575 states:

> "Thus, market efficiency per se is not testable. It must be tested jointly with some model of equilibrium, an asset pricing model. This point, the theme of the 1970 review (Fama (1970)), says that we can only test whether information is properly reflected in prices in the context of a pricing model that defines the meaning of 'properly'."

This common belief is, in fact, not correct. We claim that, consistent with the original definition, it is possible to test market efficiency without specifying *a particular* equilibrium model. We prove this assertion below. Our claim has precedence in the literature where it is well understood that the existence of an arbitrage opportunity rejects market efficiency (see, for example, Jensen [37]). And, of course, identifying an arbitrage opportunity does not require the specification of a particular equilibrium model.

More generally, the purpose of this paper is to revisit the meaning of market efficiency to rectify various misconceptions in the literature and to develop new theorems related to market efficiency. As such, one can then better understand the implications of an efficient market for empirical testing, profitable trading strategies, and the properties of asset price processes. This analysis is facilitated by our accumulated understanding of martingale pricing methods and their application to equilibrium models (for a review see Duffie [20]).

To start, we first provide an analytic definition of an efficient market with respect to an information set that is consistent with the existing definition but independent of a particular equilibrium asset pricing model. Next, we provide two alternative characterizations of this definition that facilitate both theorem proving and empirical testing.[3] The first characterization relates to the existence of an equivalent probability measure making the normalized asset price processes martingales (sometimes called risk neutral measures). The second characterization relates to no arbitrage (in the sense of No Free Lunch with Vanishing Risk (NFLVR)) and No Dominance (ND). This latter characterization formalizes the notion that an efficient market has "no profitable" trading strategies (see Jensen [37]).

These two characterizations enable us to obtain some new insights and to prove some new theorems regarding efficient markets. First we show that to test for an efficient market, one only needs to show that there are no arbitrage opportunities nor dominated securities with respect to an information set. These tests are both necessary and sufficient. Surprisingly, when restricted to discrete trading economies, we show that market efficiency is in fact equivalent only to the notion of no arbitrage (NFLVR). This is especially relevant because most of the existing empirical studies of market efficiency are based on discrete time models (see Fama [22],[23],[24], Jensen [37] for reviews). Because such empirical tests do not require the specification of a particular equilibrium model, this proves our claim that market efficiency can be tested without the joint model hypothesis.

[3] This is analogous to Delbaen and Schachermayer [15] providing a rigorous definition of no arbitrage as No Free Lunch with Vanishing Risk (NFLVR) and the resulting alternative characterization of NFLVR in terms of local martingale measures.

With respect to different information sets, we study information expansion and reduction with respect to market efficiency. As is well known in the literature, we show that information reduction is consistent with market efficiency, but information expansion may not be. If the market is semi-strong form efficient, then it is weak-form efficient; but, if the market is semi-strong form efficient, it need not be strong-form efficient. Theorems and examples illustrate these statements. With respect to information expansion, we also study the question: if the market is semi-strong form efficient and it is impossible to produce arbitrage in the sense of NFLVR with respect to inside information, then is the market strong-form efficient? In general the answer is no, but we provide sufficient conditions for its validity—if the market is either: (i) discrete time, (ii) complete, or (iii) the H-hypothesis holds. The H-hypothesis is a mathematical condition often used in the area of credit risk pricing and hedging (see Elliott, Jeanblanc and Yor [21] and Bielecki and Rutkowski [3]). Our analysis thus provides an economic interpretation of the H-hypothesis relating to market efficiency.

We also study the conditions imposed by market efficiency on an asset price process beyond those imposed by no arbitrage (NFLVR) alone. These insights have two uses. First, they provide an alternative method for testing market efficiency based on a joint hypothesis. Here the joint hypothesis is the specification of a particular stochastic process for asset prices. This additional hypothesis is testable independently of market efficiency. And, an efficient market is a nested subclass—the price process supports efficiency if its parameters are in a particular subset and it is inefficient otherwise. In contrast, the classical joint hypothesis—specifying a particular equilibrium model—is not independently testable. The equilibrium model and efficiency are both accepted or rejected in unison. Second, these insights are also useful when one wants to impose more structure on the economy than just NFLVR to capture market wide conditions related to aggregate supply equalling aggregate demand. This additional structure has already proven relevant in the study of asset price bubbles (see Jarrow, Protter and Shimbo [35], [36]). For pricing and hedging purposes, we illustrate the additional restrictions imposed by an efficient market on various stochastic volatility models that are useful for pricing equity and index options.

An outline for this paper is as follows. Section 2 introduces the model structure while section 3 defines an efficient market and proves various characterization theorems. Section 4 discusses different information sets, section 5 presents some market efficient price processes, and section 6 concludes.

2 The Model

We consider a continuous time and continuous trading economy on an infinite horizon. There are a finite number of traders in the economy. Securities markets are assumed to be competitive and frictionless.

2.1 The Market

We are given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ on $[0, \infty)$ that satisfies the usual conditions. P is the statistical probability measure. The traded securities consist of a locally riskless money market account together with d risky securities whose market prices at time t, given in units of the money market account, are $S(t) = (S^1(t), \ldots, S^d(t))$. We let a security S^0 corresponds to the locally riskless money market account with $S^0(t) \equiv 1$. To simplify the presentation we assume that the securities have no cash flows. We also make the following assumption:

$$S^i(t) \geq 0 \text{ a.s. for all } t \text{ and } i = 1, ..., d.$$

$S = (S(t))_{t \geq 0}$ denotes a vector stochastic process, and we let $\mathbb{F}^S$ denote the natural filtration of S, made right-continuous and augmented with the P-null sets. The process S is assumed to be a (not necessarily locally bounded) semimartingale with respect to $\mathbb{F}^S$. We assume that $\mathbb{F}$ contains $\mathbb{F}^S$ and that S is a semimartingale with respect to $\mathbb{F}$. Although we do not require that $\mathcal{F}_0$ be P-trivial, we do assume that S_0 is a.s. constant.

For a given filtration $\mathbb{F}$, we refer to the pair $(\mathbb{F}, S)$ as a *market*.

2.2 Trading Strategies

The economy is populated by a finite number of investors each of whom have the beliefs P and the information filtration $\mathbb{F}$. Due to the competitive markets assumption, traders act as price takers. Given frictionless markets (no transaction costs nor restrictions on trade), the trading strategies available to an investor are modeled by $\mathbb{F}$-*admissible strategies* H. That is, H is an $\mathbb{F}$ predictable and S-integrable process which is $(\mathbb{F}, a)$-*admissible* for some $a \in \mathbb{R}$, meaning that $H \cdot S \geq -a$. Here,

$$(H \cdot S)_t = \sum_{i=0}^{d} \int_0^t H^i(s) dS^i(s)$$

corresponds to a vector stochastic integral, see Protter [49] and Jacod [31]. We use the convention that $(H \cdot S)_0 = 0$.

We require that the admissible trading strategies be *self-financing*, meaning that there are no cash flows generated by the trading strategy. That is, letting $V(t) = \sum_{i=0}^{d} H^i(t) S^i(t)$ denote the time t value of the trading strategy, the self-financing condition is that $V(t) = V(0) + (H \cdot S)_t$ for all t. A variant of the self-financing condition will be discussed later in the context of endowment and consumption streams.

2.3 No Arbitrage (NFLVR)

Our no arbitrage condition is the classical No Free Lunch with Vanishing Risk (NFLVR) due to Delbaen and Schachermayer [15], [17]. NFLVR means that there is no sequence $f_n = (H^n \cdot S)_\infty$, where each H^n is admissible and $(H^n \cdot S)_\infty$ exists, such that $\| \max(-f_n, 0) \|_\infty \to 0$ and $f_n \to f$ a.s. for some $f \geq 0$ with $P(f > 0) > 0$. In our context, we will need to impose NFLVR on specific time intervals. We therefore make the following definition (note that taking $T = \infty$ yields the usual definition of NFLVR).

Definition 1 *A market $(\mathbb{F}, S)$ satisfies NFLVR on [0,T] if the stopped process S^T, together with the filtration $\mathbb{F}$, satisfies NFLVR.*

The Fundamental Theorem of Asset Pricing (see Delbaen and Schachermayer [15], [17]) states that in our setting NFLVR is equivalent to the existence of an equivalent local martingale measure[4]. In other words, a market $(\mathbb{F}, S)$ satisfies NFLVR on $[0, T]$ if and only if the set

$$\mathcal{M}_{loc}(\mathbb{F}, S, T) = \{Q : Q \sim P \text{ and } S \text{ is an } (\mathbb{F}, Q) \text{ local martingale on } [0, T]\}$$

is non-empty. When there is no risk of confusion, we will sometimes simply write $\mathcal{M}_{loc}$, $\mathcal{M}_{loc}(\mathbb{F})$, etc.

2.4 No Dominance (ND)

The notion of No Dominance (ND) was introduced by Merton [45] to study the properties of option prices. Merton's definition can be formalized as follows.

Definition 2 (No Dominance) *the i^{th} security $S^i = (S^i(t))_{t\geq 0}$ is undominated on $[0, T]$ if there is no admissible strategy H such that*

$$S^i(0) + (H \cdot S)_T \geq S^i(T) \;\; a.s. \qquad and \qquad P\{S^i(0) + (H \cdot S)_T > S^i(T)\} > 0.$$

A market $(\mathbb{F}, S)$ satisfies ND on $[0, T]$ if each S^i, $i = 0, \ldots, n$, is undominated on $[0, T]$.

In words, ND states that it is not possible to find a trading strategy that generates a set of payoffs at time T that dominate the payoffs to any traded security. ND has been used recently in the literature by Jarrow, Protter and Shimbo [35], [36] for the study of asset price bubbles. Moreover, a closely related notion known as "Relative Arbitrage" has been recently studied by Fernholz, Karatzas, Kardaras, Ruf, and others; see for instance [26], [25] and [51].

Notice that the above definition also makes sense for $T = \infty$. The reason is that $(H \cdot S)_\infty$ exists for every admissible H, so in particular $S^i(0) + (H^i \cdot S)_\infty = S^i(\infty)$ exists for every i, where H^i is given by

$$H^i = (0, \ldots, 0, 1, 0, \ldots, 0), \tag{1}$$

with the one in position i. This shows that ND on $[0, \infty]$ is a well-defined notion in the presence of NFLVR. In addition, we point out that if S^i is undominated on $[0, T]$, it is also undominated at all earlier times $T' < T$. Indeed, if there were a dominating strategy H, one could apply the strategy $K(t) = H(t)\mathbf{1}_{\{t\leq T'\}} + H^i(t)\mathbf{1}_{\{t>T'\}}$ where H^i is as in (1). This corresponds to holding one unit of asset i up to the time horizon. The nonnegativity of S^i ensures that H^i is admissible. The strategy K satisfies

$$S^i(0) + (K \cdot S)_T = S^i(T) + S^i(0) + (H \cdot S)_{T'} - S^i(T') \geq S^i(T),$$

with positive probability of having a strict inequality. But, this is impossible since S^i is undominated on $[0, T]$.

[4]Notice that we do not have to distinguish between local martingales and sigma martingales since prices are nonnegative. This follows from the definition of a sigma martingale and the Ansel-Stricker theorem.

NFLVR and ND are distinct conditions, but both imply the simpler No Arbitrage (NA) condition: there can be no admissible strategy H such that

$$(H \cdot S)_T \geq 0 \text{ a.s.} \qquad \text{and} \qquad P\{(H \cdot S)_T > 0\} > 0.$$

Indeed, since ND in particular implies that $S^0 \equiv 1$ is undominated, it follows that ND implies NA. And, it has been shown by Delbaen and Schachermayer [15] that a market $(\mathbb{F}, S)$ satisfies NFLVR if and only if it satisfies NA together with the condition that the set of payoffs of 1-admissible strategies with bounded support is bounded in probability.

2.5 Maximal Trading Strategies

Essential in proving many of our results in the notion of maximal trading strategies introduced by Delbaen and Schachermayer [17].

Definition 3 (Maximal Strategies) *A process H is called* $\mathbb{F}$-maximal on $[0, T]$ *if it is $\mathbb{F}$-admissible and for every $\mathbb{F}$-admissible strategy K such that $(K \cdot S)_T \geq (H \cdot S)_T$, it is true that $(K \cdot S)_T = (H \cdot S)_T$.*

If the filtration and/or the time horizon is clear from the context, we drop these qualifiers and simply call H *maximal*.

To understand the meaning of a maximal trading strategy H, one first fixes a time T payout generated by a trading strategy $(H \cdot S)_T$. Then, a maximal admissible trading strategy has the largest such fixed payoff possible starting at time 0 with zero investment. In terms of maximality, the No Dominance (ND) condition can be phrased as requiring that all the strategies H^i in (1) are maximal.

We need two results from Delbaen and Schachermayer [17] concerning maximal strategies.

Lemma 1 *If S is a positive $\mathbb{F}$ semimartingale that satisfies NFLVR with respect to $\mathbb{F}$, then for any $\mathbb{F}$-admissible strategy H the following are equivalent:*

(*i*) *H is $\mathbb{F}$-maximal on $[0, T]$.*

(*ii*) *There is $Q \in \mathcal{M}_{loc}(\mathbb{F})$ such that $H \cdot S$ is an $(\mathbb{F}, Q)$ martingale on $[0, T]$.*

(*iii*) *There is $Q \in \mathcal{M}_{loc}(\mathbb{F})$ such that $E_Q(H \cdot S)_T = 0$.*

Proof. See [17], Theorem 5.12., while keeping in mind that local martingale measures and sigma martingale measures coincide in our setting where S is nonnegative. ■

Lemma 2 *Finite sums of maximal strategies are again maximal.*

Proof. This follows from Theorem 2.14 in [16], which is stated for the case where S is locally bounded. However, an examination of the proof of this theorem, and the results that it relies on (Lemma 2.11, Proposition 2.12 and Proposition 2.13 in the same reference) show that the local boundedness is never used. ■

2.6 An Economy

We consider a pure exchange economy on a finite horizon $[0, T]$. An economy consists of a market $(\mathbb{F}, S)$ and a finite number of investors $(k = 1, ..., K)$ characterized by their beliefs, information, preferences, and endowments.

We let α^i denote the aggregate net supply of the i^{th} security. It is assumed that each α^i is non-random and constant over time, with $\alpha^0 = 0$ and $\alpha^i > 0$ for $i = 1, \ldots, d$.

There is a single consumption good that is perishable. The price of the consumption good, in units of the money market account, is denoted $\psi = \{\psi(t) : 0 \leq t \leq T\}$. We assume that $\psi(t)$ is strictly positive.

Each investor solves an optimization problem where he seeks to maximize utility from consumption. In Karatzas and Žitković [43], the optimizing agent receives endowments and consumes his wealth continuously through time, using a general incomplete semimartingale financial market to finance his consumption. The utility structure is very general, allowing among other things for state dependent utility functions. We adopt a similar setup. Let μ be the probability measure on $[0, T]$ such that $\mu(\{T\}) > 0$. Two canonical examples are $\mu([0, T)) = 0$, $\mu(\{T\}) = 1$, which corresponds to utility from terminal consumption only, and

$$\mu(dt) = \frac{1}{2T}dt + \frac{1}{2}\delta_{\{T\}}(dt),$$

which is diffuse on $[0, T)$ and has an atom $\{T\}$. This corresponds to utility from continuous consumption over $[0, T)$ and a bulk consumption at T. The use of the measure μ simplifies the notation by allowing us to treat utility from intermediate and final consumption within a single framework.

The k^{th} investor is characterized by the following quantities.

- *Beliefs and information* $(P, \mathbb{F})$. We assume that all investors have the same beliefs and information.

- A time dependent *utility function* $U_k : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ such that for each t in the support of μ, the function $U_k(t, \cdot)$ is concave and strictly increasing. The utility that agent k derives from consuming $c(t)\mu(dt)$ at each time $t \leq T$ is

$$\mathcal{U}_k(c) = E\left(\int_0^T U_k(t, c(t))\mu(dt)\right).$$

  Since $\mu(\{T\}) > 0$, the utility is strictly increasing in the final consumption $c(T)$.

- *Initial wealth* x_k. Given a trading strategy $H = (H^1, \ldots, H^d)$, the investor will be required to choose his initial holding $H^0(0)$ in the money market account such that

$$x_k = H^0(0) + \sum_{i=1}^{d} H^i(0)S^i(0). \tag{2}$$

- An *endowment stream* $\epsilon_k(t)$, $t < T$ of the commodity. This means that the investors receive $\epsilon_k(t)\mu(dt)$ units of the commodity at time $t \leq T$. The cumulative endowment of the k^{th} investor, in units of the money market account, is given by

$$\mathcal{E}_k(t) = \int_0^t \psi(s)\epsilon_k(s)\mu(ds).$$

The setup is quite general and includes most formulations studied in the utility maximization literature. In Kramkov and Schachermayer [44], utility from terminal wealth in incomplete markets is considered, in which case $\psi \equiv 1$, $\mu(\{T\}) = 1$, and $\epsilon_k \equiv 0$. These results are extended in Cvitanić, Schachermayer and Wang [10] to the case of random endowments, relaxing the condition $\epsilon_k \equiv 0$. In Karatzas and Žitković [43], the optimizing agent receives endowments and consumes his wealth continuously through time, so $\mu([0,T))$ is no longer zero. In fact, $\mu([0,t]) > 0$ is assumed for each $t < T$. All the above papers make additional assumptions on the utility function $U_k(t,\cdot)$ for some or all of their results. In particular, it is assumed that for each t in the support of μ, the function $U_k(t,\cdot)$ is strictly concave, strictly increasing, continuously differentiable, and satisfies the Inada conditions: $\partial_2 U_k(t,0+) = \infty$ and $\partial_2 U_k(t,\infty) = 0$. Moreover, a condition that figures prominently is *reasonably asymptotic elasticity* condition. In Kramkov and Schachermayer [44] and Cvitanić, Schachermayer and Wang [10] it takes the form

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1,$$

where $U(x) = U(T,x)$. In Karatzas and Žitković [43], a uniform in time version of this condition is used, together with additional regularity conditions. It is also possible to relax other aspects of the utility structure. In Karatzas and Žitković [43], the utility function is allowed to evolve stochastically in a progressively measurable way. This would require boundedness assumptions on $\psi(t)$, see Example 3.4 in Karatzas and Žitković [43]. Finally, we mention Biagini and Frittelli [2], where utilities defined on $\mathbb{R}$ are considered.

Each investor chooses a consumption plan $\{c_k(t) : 0 \le t \le T\}$ with $c_k(t) \ge 0$, and a trading strategy in the money market account, H_k^0, and the risky securities, $H_k = (H_k^1, \ldots, H_k^d)$. The investor's wealth $W_k(t)$ at time t is

$$W_k(t) = H_k^0(t) + \sum_{i=1}^{d} H_k^i(t) S^i(t),$$

and the holdings $H_k^0(t)$ of the money market account must be chosen so that the strategy is *self-financing*, i.e.,

$$W_k(t) = x_k + \mathcal{E}_k(t) + \int_0^t H_k(u)dS(u) - C_k(t)$$

where

$$C_k(t) = \int_0^t \psi(s)c_k(s)\mu(ds)$$

is the value of cumulative consumption. Note that the self-financing condition guarantees that (2) holds.

At time T, the investors' financial holdings are transformed into units of the consumption good, which can be consumed. That is, at time T the k^{th} investor receives a liquidating dividend of

$$\frac{H_k^0(T) + \sum_{i=1}^d H_k^i(T)S^i(T)}{\psi(T)},$$

in units of the consumption good.

A pair (c_k, H_k) is called *admissible* if c_k is progressively measurable, H_k is admissible in the usual sense, and it generates a wealth process W_k with nonnegative terminal wealth, $W_k(T) \geq 0$. The consumption rate process c_k is called admissible if there exists H_k such that (c_k, H_k) is admissible.

Investor k solves the following optimization problem:

The Investor's Problem: *To maximize $U_k(c)$ over all admissible consumption plans $c = \{c(t) : 0 \leq t \leq T\}$. For fixed endowments we write*

$$u_k(x) = \sup\{\mathcal{U}_k(c) : \ c \text{ is admissible, } x_k = x\}$$

In the utility maximization literature the existence of an optimal solution has been established under a wide range of assumptions. One common condition is to require $u_k(x) < \infty$ for some $x > 0$, together with the existence of an equivalent local martingale measure. In our setting, we directly assume the existence of an optimal solution to the investor's problem. This is a powerful assumption with several important consequences.

Lemma 3 *Assume that for some $x > 0$, the investor's problem has an optimal solution with a finite optimal value. Let $(\hat{c}, \hat{H})$ be an admissible pair such that $\hat{c}$ achieves the optimum. Then $\hat{H}$ is a maximal strategy.*

Proof. If $\hat{H}$ is not maximal, there is an admissible strategy J such that $\int_0^T J(t)dS(t) \geq \int_0^T \hat{H}(t)dS(t)$, with strict inequality with positive probability. Hence this strategy supports the same consumption $\hat{c}(t)$ for $t < T$, and the additional final consumption

$$\hat{c}(T) + \frac{\int_0^T J(t)dS(t) - \int_0^T H(t)dS(t)}{\mu(\{T\})},$$

which is nonnegative and strictly positive with positive probability. This strictly improves the utility of the investor, contradicting the optimality of $(\hat{c}, \hat{H})$. ■

We note that as in Karatzas and Žitković [43] we may restrict the investors' portfolio choices to strategies $H_t \in \mathcal{K}$ a.s. for all $t \in [0, T]$ where $\mathcal{K}$ is a convex cone describing trading restrictions, such as a short sales prohibition. The proof of Lemma 3 still goes through, but maximality now refers to the restricted set of admissible strategies.

Lemma 4 *Assume that for some $x > 0$, the investor's problem has an optimal solution with finite optimal value. Then S satisfies NFLVR. Consequently, $\mathcal{M}_{loc}$ is non-empty.*

Proof. By a well-known characterization of NFLVR, it suffices to show that: (a) NA is satisfied, and (b) the set $\mathcal{K} = \left\{ \int_0^T H(s)dS(s) : H \text{ is 1-admissible}\right\}$ is bounded in L^0, see [15], Corollary 3.9.

Let $(\hat{c}, \hat{H})$ be an optimal consumption-investment plan. Suppose first NA fails, and let J be an arbitrage strategy. The strategy $\tilde{H} = \hat{H} + J$ is then admissible, and with $\tilde{X}_T = \int_0^T \tilde{H}(t)dS(t)$ and $\hat{X}_T = \int_0^T \hat{H}(t)dt$, we have $\tilde{X}_T \geq \hat{X}_T$ and $P(\tilde{X}_T > \hat{X}_T) > 0$. Hence $\hat{H}$ is not maximal, which is impossible by Lemma 3.

Next, the fact that the set $\mathcal{K}$ is bounded in L^0 follows from a straightforward adaptation of the proof of Proposition 4.19 in [40]. The argument goes through almost unchanged as soon as we have established that $u(\cdot)$ is concave. For this, choose arbitrary $x^i > 0$ for $i = 1, 2$ and $\lambda \in [0, 1]$, and set $x^0 = \lambda x^1 + (1-\lambda)x^2$. There are sequences $\{c_i^n\}_{n\in\mathbb{N}}$, $i = 1, 2$, of consumption plans such that c_n^i is admissible given initial capital x^i, and

$$u(x^i) = \lim_{n\to\infty} E\Big[\int_0^T U(t, c_n^i(t))\mu(dt)\Big].$$

Now, $c_n^0 = \lambda c_n^1 + (1-\lambda)c_n^2$ is admissible with initial capital x^0. Hence, due to the concavity of $U(t, \cdot)$ for $t \in [0, T]$, we get

$$u(x^0) \geq \limsup_{n\to\infty} E\Big[\int_0^T U(t, c_n^0(t))\mu(dt)\Big] \geq \lambda u(x^1) + (1-\lambda)u(x^2).$$

Thus $u(\cdot)$ is concave, as claimed. ∎

This lemma is the formalization of the well-known result that the existence of an investor's optimal consumption choice implies that there are no arbitrage opportunities.

An *economy* is defined by the collection $(P, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$.

2.7 An Equilibrium

This section defines a market equilibrium and explores its implications. Given an economy $(P, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$, an economic equilibrium determines the price processes (ψ, S) by equating aggregate supply equal to aggregate demand. This is formalized in the following definition.

Definition 4 (Equilibrium) *Given an economy $(P, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$, a consumption good price index ψ, financial asset prices $S = (S^0, S^1, \ldots, S^d)$, and investor consumption-investment plans $(\hat{c}_k, \hat{H}_k)$ for $k = 1, \ldots, K$, the pair (ψ, S) is an equilibrium price process if for all $0 \leq t \leq T$ a. e. P,*

(i) securities markets clear:

$$\sum_{k=1}^K \hat{H}_k^i(t) = \alpha^i, \qquad i = 0, \ldots, d;$$

(ii) commodity markets clear:

$$\sum_{k=1}^{K} \hat{c}_k(t) = \sum_{k=1}^{K} \epsilon_k(t);$$

(iii) investors' choices are optimal: $(\hat{c}_k, \hat{H}_k)$ *solves the* k^{th} *investor's utility maximization problem and the optimal value is finite.*

Such an equilibrium is sometimes called an Arrow-Radner equilibrium. Sufficient conditions for the existence of such an equilibrium can be found in Duffie [19], Karatzas, Lehoczky and Shreve [42], Dana and Pontier [14], Dana [12], [13], and Žitković [55].

We now establish some properties that must hold in an economic equilibrium. Notice that NFLVR always holds in equilibrium as a consequence of Lemma 4.

Lemma 5 *Suppose an equilibrium is given. Then holding the market portfolio is a maximal strategy, i.e.* $H = (H^1, \dots, H^d)$ *given by*

$$H^i(t) \equiv \alpha^i, \qquad i = 1, \dots, d$$

is maximal.

Proof. By Lemma 4, $\mathcal{M}_{loc} \neq \emptyset$. Furthermore, Lemma 3 implies that each $\hat{H}_k$ is maximal. By Lemma 2, their sum $H = \hat{H}_0 + \cdots + \hat{H}_K$ is also maximal. But the clearing condition for the securities markets implies that $H^i \equiv \alpha^i$ for each $i = 1, \dots, d$. ∎

The next result shows that buying and holding assets in positive net supply is also a maximal strategy.

Lemma 6 *Suppose an equilibrium is given. Then, for each fixed* $i \in \{0, 1, \dots, d\}$*, the strategy* $H = (H^0, \dots, H^d)$ *given by*

$$\begin{cases} H^i \equiv 1 \\ \\ H^j \equiv 0, \quad j \neq i \end{cases}$$

is maximal, i.e. ND holds.

Proof. By Lemma 4, NFLVR and hence NA holds, so the claim is true for $i = 0$. Suppose $i \in \{1, \dots, d\}$ and let $\tilde{H}$ be the market portfolio from Lemma 5, multiplied by a factor $1/\alpha^i$. This is well-defined since $\alpha^i > 0$, and $\tilde{H}$ is still maximal because maximality is not affected by positive scaling. By Lemma 5 and Lemma 1, there is a probability $Q \in \mathcal{M}_{loc}$ under which $\int \tilde{H} dS$ becomes a martingale. Due to the nonnegativity of asset prices,

$$\sum_{i=1}^{d} S^i(0) + \int \tilde{H} dS = S^i + \sum_{i \neq j} \frac{\alpha^j}{\alpha^i} S^j \geq S^i.$$

Hence under Q, S^i is a nonnegative local martingale dominated by a true martingale, and therefore itself a true martingale. Another application of Lemma 1 gives the maximality of H. ∎

As presented, our equilibrium is for an economy with symmetric information. An interesting extension is the asymmetric information case, where all traders share the same beliefs P but have different information sets represented by the filtrations $\mathbb{F}^k$. Furthermore, the market filtration $\mathbb{F} = \bigcap_k \mathbb{F}^k$ consists of the information that is available to all traders. In the investor's optimization problem, $\mathbb{F}^k$ replaces $\mathbb{F}$. Hence, the k^{th} investor's consumption and portfolio choices (c_k, H_k) are admissible with respect to $\mathbb{F}^k$. His optimal strategy $\hat{H}_k$ will be $\mathbb{F}^k$-maximal, and since $\mathbb{F} \subset \mathbb{F}^k$ it is intuitively clear that no $\mathbb{F}$-admissible strategy can dominate $\hat{H}_k$. However, there are technical issues related to the invariance of stochastic integrals as the filtration changes, which we leave for future research.

All else remains the same, with a market still being the pair $(\mathbb{F}, S)$. The definition of an equilibrium is unchanged with equilibrium prices reflecting the market clearing conditions (i) and (ii), and investors' decisions being optimal (iii), with the changed measurability requirements. When discussing NFLVR and ND, the market information set $\mathbb{F}$ is the relevant one. This asymmetric information extension relates our equilibrium notion to that of a Rational Expectations Equilibrium (REE), see Jordan and Radner [39] and Admati [1] for reviews. Since $\mathbb{F}^S \subset \mathbb{F} \subset \mathbb{F}^k$, an investor's decisions are conditioned on the information revealed by prices. An equilibrium price process (ψ, S), therefore, confirms the investors' beliefs conditioned on $\mathbb{F}^S$.

3 Market Efficiency

This section defines an efficient market and provides two equivalent characterizations that are useful for empirical testing and theorem proving.

3.1 Definition

As discussed in the introduction, it is commonly believed that to test market efficiency, one needs to assume a particular equilibrium model in order to investigate its implications relating to the properties of the price process or the existence of abnormal trading profits. Both of these implications are derived from the martingale properties of the equilibrium price processes and they were first discovered by Samuelson [52]. If these implications are violated in the empirical study, then efficiency is rejected. In fact, Jensen [37], p. 96 in his review of the empirical literature uses these necessary conditions as the definition of an efficient market:

> "A market is efficient with respect to information set θ_t if it is impossible to make economic profits by trading on the basis of information set θ_t. By economic profits, we mean the risk adjusted returns net of all costs. Application of the zero profit condition to speculative markets under the assumption of zero storage costs and zero transactions costs gives us the result that asset prices (after the adjustment for required returns) will behave as a martingale with respect to the information set θ_t."

Consistent with the intent of these definitions, we provide a model independent and rigorous definition of an efficient market that has content (to be shown) and can be empirically tested (also to be shown), i.e.

Definition 5 *A market* $(\mathbb{F}, S)$ *is called* efficient on $[0,T]$ *with respect to* $\mathbb{F}$ *if there exists a consumption good price index* ψ *and an economy* $(P, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$ *for which* (ψ, S) *is an equilibrium price process* S *on* $[0,T]$*.*

If this holds for every $T < \infty$*, the market is called* efficient *with respect to* $\mathbb{F}$*.*

This definition says that a market $(\mathbb{F}, S)$ is efficient with respect to $\mathbb{F}$ if there exists an economy whose equilibrium price process is consistent with S.[5]

3.2 Characterization Theorems

This section gives several different characterizations of an efficient market. Our first characterization relates an efficiency on $[0,T]$ to the economic notions of ND and NFLVR. The second gives a description in terms of equivalent martingale measures. The following theorem is the main result of this section.

Theorem 1 (Characterization of efficiency) *Let* $(\mathbb{F}, S)$ *be a market. The following statements are equivalent.*

(i) $(\mathbb{F}, S)$ *is efficient on* $[0,T]$*.*

(ii) $(\mathbb{F}, S)$ *satisfies both NFLVR and ND on* $[0,T]$*.*

(iii) There exists a probability Q*, equivalent to* P*, such that* S *is an* $(\mathbb{F}, Q)$ *martingale on* $[0,T]$*. That is,* $\mathcal{M}(\mathbb{F}, S, T) \neq \emptyset$*.*

Proof. $(i) \Longrightarrow (ii)$: If $(\mathbb{F}, S)$ is efficient on $[0,T]$, there is a consumption good price index ψ and an economy $(P, \mathbb{F}, \{\epsilon_k\}_{k=1}^K, \{U_k\}_{k=1}^K)$ such that (ψ, S) is an equilibrium price process. Hence by Lemma 4 and Lemma 6, both NFLVR and ND hold.

$(ii) \Longrightarrow (iii)$: If $(\mathbb{F}, S)$ satisfies ND and NFLVR, then all the strategies H^i in (1) are maximal. By Lemma 2, $H = H^1 + \ldots + H^n = (1, \ldots, 1)$ is then also maximal. Lemma 1 thus implies that there is $Q \in \mathcal{M}(\mathbb{F})$ turning

$$H \cdot S = (S^1 - S^1(0)) + \ldots + (S^n - S^n(0))$$

into a martingale. Using the nonnegativity of S, we see that each nonnegative Q local martingale S^i is dominated by a martingale, and is therefore itself a martingale.

$(iii) \Longrightarrow (i)$: Assume that there exists an equivalent martingale measure Q. We need to construct an equilibrium supporting the price process S. Let all investors have logarithmic utilities,

$$U_k(x) = \begin{cases} \ln x, & x > 0 \\ -\infty, & x \leq 0 \end{cases}$$

[5] In the context of an asymmetric information economy, a fully revealing REE is an equilibrium price process (ψ, S) such that $\mathbb{F}^S = \bigvee_{k=1}^K \mathbb{F}^k$, i.e. all private information is reflected in the market price process. Since also $\mathbb{F}^S \subset \mathbb{F}^k$, it follows that $\mathbb{F}^S = \mathbb{F}^k$ for each k. That is, all investors share the same information set, namely the information contained in the prices. A partially revealing REE is an equilibrium price process where this is not the case. A fully revealing REE corresponds to strong-form market efficiency, while a partially revealing REE corresponds to weak-form efficiency.

for each k, and suppose they only derive utility from terminal consumption, i.e. $\mu(\{T\}) = 1$. Set $\psi(t) \equiv 1$ and assume that the endowment streams ϵ_k are identically zero—then the investors only receive utility from the liquidating dividend.

Next, suppose that the investor beliefs are given by an equivalent probability P^*, which we define via

$$\frac{dP^*}{dQ} = Z(T),$$

where

$$Z(t) = \frac{\alpha^1 S^1(t) + \cdots + \alpha^d S^d(t)}{\alpha^1 + \cdots + \alpha^d},$$

which is a Q-martingale. The k^{th} investor's optimization problem is then

$$\sup\Big\{E_{P^*}[U_k(X(T))] \;:\; X(T) = x_k + \int_0^T H(s)dS(s),\ H \text{ admissible}\Big\}.$$

Since $U_k(x) = -\infty$ for $x \leq 0$, we may restrict attention to strategies for which $X(T) > 0$. Then, due to the supermartingale property of $X = x_k + \int H(t)dS(t)$ under Q, $X(t) \geq E_Q(X(T) \mid \mathcal{F}_t) \geq 0$ for all $t \leq T$. Hence, in fact, we only need to consider x_k-admissible strategies.

Then, with the preferences and beliefs described above, the optimal strategy for each investor is to invest his initial wealth in the market portfolio until the time horizon T. As a consequence, there is an equilibrium supporting the market $(\mathbb{F}, S)$. Indeed, to prove this, let H be any 1-admissible strategy, and set $X = 1 + \int HdS$. Jensen's inequality, the definition of P^*, and the supermartingale property of X under Q yield

$$E_{P^*}\Big[\ln\frac{X(T)}{Z(T)}\Big] \leq \ln E_{P^*}\Big[\frac{1}{Z(T)}X(T)] = \ln E_Q[X(T)] \leq \ln 1 = 0.$$

Hence

$$E_{P^*}[\ln Z(T)\Big] = E_{P^*}[\ln X(T)] - E_{P^*}\Big[\ln\frac{X(T)}{Z(T)}\Big] \geq E_{P^*}[\ln X(T)].$$

A process $\tilde{X}$ is the gains process of an x_k-admissible strategy if and only if it is of the form $x_k X$ for X as above. Since the above yields

$$\begin{aligned} E_{P^*}[U_k(x_k Z(T))] &= \ln x_k + E_{P^*}[\ln Z(T)] \\ &\geq \ln x_k + E_{P^*}[\ln X(T)] = E_{P^*}[U_k(x_k X(T))], \end{aligned}$$

we conclude that investing x_k in the market portfolio and holding until time T is optimal.

It is now straightforward to verify that we have an equilibrium. With preferences as described above, the k^{th} investor's holdings in the i^{th} asset at time t is given by

$$\hat{H}_k^i(t) = x_k \frac{\alpha^i}{\alpha^1 S^1(0) + \cdots + \alpha^d S^d(0)}.$$

Summing over k and using that $\sum_{k=1}^K x_k = \alpha^1 S^1(0) + \cdots + \alpha^d S^d(0)$ shows that the securities markets clear. The commodity markets also clear, since there is no intermediate consumption or endowments. This concludes the proof. ■

Characterization (iii) formalizes the connection between martingales and efficiency as first noted by Samuelson [52] and Fama [22]. As pointed out previously, by the Fundamental Theorem of Asset Pricing, NFLVR on $[0,T]$ implies that $\mathcal{M}_{loc}(\mathbb{F},S,T) \neq \emptyset$. The efficiency condition is stronger. It requires that $\mathcal{M}(\mathbb{F},S,T) \neq \emptyset$ where

$$\mathcal{M}(\mathbb{F},S,T) = \{Q \sim P : S \text{ is an } (\mathbb{F},Q) \text{ martingale on } [0,T]\}.$$

The set $\mathcal{M}(\mathbb{F},S,T)$ can equivalently be described as consisting of the equivalent measures that turn S into a uniformly integrable martingale on $[0,T]$. When there is no risk of confusion we write $\mathcal{M}$, $\mathcal{M}(\mathbb{F})$, etc.

Consistent with this observation, there exist markets that satisfy NFLVR but are not efficient. An example is any complete market with a price bubble, see Jarrow, Protter and Shimbo [35]. To see this, consider the following simple economy consisting of only two traded assets, the money market account and S^1. Let S^1 be an inverse Bessel process[6]. Then $\mathcal{M}_{loc}$ consists of a single element under which S is a strict local martingale (i.e. a local martingale that is not a martingale), and hence $\mathcal{M} = \emptyset$. Theorem 1 then shows that this market, where we can take $\mathbb{F} = \mathbb{F}^S$, is not efficient.

The alternative characterization of efficiency in terms of ND and NFLVR makes precise the meaning of "no economic profits" in the definition of an efficient market as given by Jensen [37], p. 96 and quoted above. "No economic profits" means NFLVR and ND. As stated, it is self-evident that the notions of NFLVR and ND are independent of any particular equilibrium model; they must be satisfied by all such equilibrium models. It is this characterization that facilitates empirical tests of market efficiency that are independent of the joint model hypothesis.

Indeed, given any market $(\mathbb{F},S)$, to disprove efficiency one just needs to identify an arbitrage opportunity (FLVR) or a dominating trading strategy. Conversely, if one can show that no such strategies exist, then the market is efficient. To show that no such strategies exist, one can use Theorem 1, and show that such a martingale probability Q exists. Given a specification for the stochastic process S, an empirical investigation of the process's parameters could confirm or reject this possibility. In contrast to the classical joint hypothesis test of an efficient market, this alternative provides a test of market efficiency where the additional hypothesis can be independently validated (see section 5 below).

This theorem also helps us to understand the relationship between an efficient market and asset price bubbles. As shown in Jarrow, Protter and Shimbo [35], [36], a complete market that is efficient (satisfies both NFLVR and ND) has no price bubbles. However, they provide numerous examples of efficient but incomplete markets that contain price bubbles. Hence, there is a weak relationship between market efficiency and the non-existence of asset price bubbles, the link is the notion of a complete market.

Our second theorem deals with the case where $(\mathbb{F},S)$ is *efficient with respect to* $\mathbb{F}$, i.e. where efficiency on $[0,T]$ holds for every finite T (see Definition 5).

Theorem 2 *The market* $(\mathbb{F},S)$ *is efficient if and only if there is a family of probabilities* $\{Q_t\}_{t\geq 0}$*, where* Q_t *is defined on* $\mathcal{F}_t$*, such that*

[6] The inverse Bessel process can be defined as $1/\|B\|$, where B is a three-dimensional Brownian motion starting from $(1,0,0)$. See [8] for details.

(*i*) $Q_t = Q_s$ *on* $\mathcal{F}_s$ *for all* $s < t$,

(*ii*) $Q_t \sim P$ *on* $\mathcal{F}_t$ *and* S *is a* $(\mathbb{F}, Q_t)$ *martingale on* $[0, t]$.

Proof. Sufficiency follows by considering Q_T and applying Theorem 1 to $(\mathbb{F}, S)$ restricted to $[0, T]$. For necessity, it suffices to construct measures Q^n, $n \in \mathbb{N}$, such that $Q^n \sim P$, $Q^{n+1} = Q^n$ on $\mathcal{F}_n$, and S is a Q^n martingale on $[0, n]$. We construct the Q^n inductively. Let $Q^0 = P$. Suppose Q^{n-1} has been constructed, and choose $\tilde{Q}^n$, equivalent to P, such that S becomes a uniformly integrable martingale on $[0, n]$. Such a measure exists due to the hypothesis and Theorem 1. Let $Z_t^{n-1} = E_P(\frac{dQ^{n-1}}{dP} \mid \mathcal{F}_t)$ and $\tilde{Z}_t^n = E_P(\frac{d\tilde{Q}^n}{dP} \mid \mathcal{F}_t)$, and define

$$Z_t^n = \begin{cases} Z_t^{n-1} & t < n-1, \\ Z_{n-1}^{n-1} \frac{\tilde{Z}_t^n}{\tilde{Z}_{n-1}^n} & t \geq n-1. \end{cases}$$

The measure Q^n given by $\frac{dQ^n}{dP} = Z_n^n$ has density process Z^n, which coincides with Z^{n-1} on $[0, n-1]$ implying that $Q^n = Q^{n-1}$ on $\mathcal{F}_{n-1}$.

It remains to check that S is a Q^n martingale on $[0, n]$, so pick $0 \leq s < t \leq n$ and $A \in \mathcal{F}_s$. First, if $t \leq n-1$, then $E_{Q^n}(\mathbf{1}_A(S_t^i - S_s^i)) = E_{Q^{n-1}}(\mathbf{1}_A(S_t^i - S_s^i)) = 0$ for each i. If instead $s \geq n-1$, then Bayes' rule yields

$$E_{Q^n}(S_t^i \mid \mathcal{F}_s) = \frac{1}{Z_s^n} E_P(Z_t^n S_t^i \mid \mathcal{F}_s) = \frac{1}{\tilde{Z}_s^n} E_P(\tilde{Z}_t^n S_t^i \mid \mathcal{F}_s) = E_{\tilde{Q}^n}(S_t^i \mid \mathcal{F}_s) = S_s^i.$$

Finally, if $s \leq n-1 \leq t$, then

$$E_{Q^n}(\mathbf{1}_A(S_t^i - S_s^i)) = E_{Q^n}(\mathbf{1}_A(S_t^i - S_{n-1}^i)) + E_{Q^n}(\mathbf{1}_A(S_{n-1}^i - S_s^i)) = 0,$$

by the two previous cases. The proof is complete. ■

Given the consistent family $\{Q_t\}_{t \geq 0}$ from Theorem 2, a natural question is if some form of the Kolmogorov extension theorem can be used to extend the Q_t to a measure on $\mathcal{F}_\infty$. This is not possible in general. The problem, which is well understood, is best explained through an example. Consider the Black-Scholes model $S_t = S_0 \exp\{W_t + (\mu - \frac{1}{2})t\}$, where $\mu > 0$ and W is Brownian motion under P. Any risk-neutral measure Q would have to be such that $B_t = W_t + \mu t$ is Brownian motion under Q. However, the measures P and Q are not equivalent: the stopping time $\tau = \inf\{t \geq 0 : B_t = 1\}$ satisfies $P(\tau = \infty) = 0$, but $Q(\tau = \infty) > 0$. Further, Q and P are not even equivalent on $\mathcal{F}_t$ for $t < \infty$, since the usual conditions imply that the P null set $\{\tau = \infty\}$ lies in $\mathcal{F}_t$.

One can resolve this problem under some additional regularity conditions on the filtration. The technique is not new, but we nonetheless state the result since it sheds some light on the role of the null sets, whose economic meaning is somewhat opaque. Let $\mathbb{F}^o = (\mathcal{F}_t^o)_{t \geq 0}$ be a filtration that is the right-continuous modification of a *standard system* $\mathbb{E} = (\mathcal{E}_t)_{t \geq 0}$. For the definition of standard system, see Parthasarathy [48], Chapter V. In our context, the most important example of a standard system is where Ω is the set of càdlàg paths that can explode in finite time, and $\mathcal{E}_t$ is generated by the coordinate process. This example is discussed by Föllmer [27] and Meyer [46]. We make the following assumption:

$$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0} \text{ is the } P\text{-completion of } \mathbb{F}^o.$$

See Jacod and Shiryaev [33], p. 2, for a description of the completion procedure. Now, there exists an $\mathbb{F}^o$ adapted process that is P-indistinguishable from S (see Jacod and Shiryaev [33], p. 10.) Let S^o be such a process. We then have the following theorem.

Theorem 3 *The market $(\mathbb{F}, S)$ is efficient for every $T < \infty$ if and only if there is a probability Q^o on $\mathcal{F}^o_\infty$ such that*

(i) $Q^o \sim P$ on $\mathcal{F}_t$ for all $t < \infty$,

(ii) S^o is an $(\mathbb{F}^o, Q^o)$ martingale.

The proof, which involves a lot of technical verification, is omitted but available from the authors upon request.

3.3 Discrete Time Markets

Most of the empirical literature testing for market efficiency utilizes discrete time markets (see Fama [22],[23],[24] and Jensen [37] for reviews). Hence, it is important to understand the characterization of market efficiency in a discrete time model. Surprisingly, we have the following theorem:

Theorem 4 *Let $(\mathbb{F}, S)$ be a market in discrete time, $t \in \{0, 1, \ldots\}$. A discrete time market $(\mathbb{F}, S)$ is efficient on $\{0, \ldots, T\}$ with respect to $\mathbb{F}$ if and only if it satisfies NFLVR on $\{0, \ldots, T\}$.*

This is an important result for interpreting the empirical finance literature that tests for market efficiency. As this theorem states, an efficient market is one that contains no arbitrage opportunities. This is both a necessary and sufficient condition for market efficiency in discrete time markets. Its use to test for an efficient market without specifying a particular equilibrium model is self-evident.

The proof of this theorem uses the following lemma, whose proof is in the Appendix.[7]

Lemma 7 *Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots,T}$ be a filtered probability space in finite discrete time. Any local martingale L that satisfies $E(|L_0|) < \infty$ and $E(L_T^-) < \infty$ is a true martingale.*

As a result of the lemma we have the following corollary, which in conjunction with Theorem 1 proves Theorem 4.

Corollary 1 *Let $(\mathbb{F}, S)$ be a market in discrete time, $t \in \{0, 1, \ldots\}$. If S is an $(\mathbb{F}, Q)$ local martingale on $\{0, \ldots, T\}$, it is in fact a martingale. Hence if (NFLVR) holds on $\{0, \ldots, T\}$, the market is automatically efficient on $\{0, \ldots, T\}$.*

[7] The second author learned this result from Martin Schweizer in a course at ETH Zurich.

Proof. The first statement is immediate from Lemma 7, which applies since each S^i is nonnegative and S_0^i is deterministic. The second statement follows from the Fundamental Theorem of Asset Pricing. ■

Notice that since NFLVR implies that a true martingale measure exists, the Dalang-Morton-Willinger (DMW) Theorem [11] lets us conclude that in discrete time, NFLVR excludes arbitrage using strategies that are not necessarily admissible. Conversely, if no such arbitrage opportunities exist, the DMW Theorem gives an equivalent martingale measure, thus showing that the market is efficient. This connection is relevant, because in discrete time the setting of the DMW Theorem is arguably more suitable than that of NFLVR.

4 Different Information Sets

In this section we study how market efficiency is affected by changes in the information sets, both information reductions and expansions. More formally we consider nested filtrations $\mathbb{F} \subset \mathbb{G}$, and study conditions under which efficiency with respect to $\mathbb{F}$ carries over to $\mathbb{G}$, and vice-versa. We work on the infinite horizon economy $[0, \infty)$, although all results remain valid for finite horizons $[0, T]$ as well. The results in this section relies crucially on the characterization of efficiency in terms of equivalent martingale measures. The corresponding analysis in the context of an equilibrium model would be much more complicated.

4.1 Filtration Reduction

If $(\mathbb{G}, S)$ is known to be efficient and we want to deduce the efficiency of $(\mathbb{F}, S)$, the analysis is particularly simple. We therefore start by treating this case. The following result is classical, see e.g. [49], Theorem I.21:

Lemma 8 *Let a filtered probability space be given. A càdlàg, adapted process M such that*

$$E(|M_\tau|) < \infty \qquad \text{and} \qquad E(M_\tau) = E(M_0)$$

for every $[0, \infty]$-valued stopping time τ is a uniformly integrable martingale.

Theorem 5 *Let S be an n-dimensional $\mathbb{G}$ semimartingale with nonnegative components and suppose that the market $(\mathbb{G}, S)$ is efficient. If $\mathbb{F} \subset \mathbb{G}$ is a filtration to which S is adapted, then S is an $\mathbb{F}$ semimartingale, and $(\mathbb{F}, S)$ is efficient.*

Proof. By Theorem 1 there is $Q \sim P$ such that S is a $(\mathbb{G}, Q)$ uniformly integrable martingale. Let τ be any $[0, \infty]$-valued $\mathbb{F}$ stopping time. It is then also a $\mathbb{G}$ stopping time, so $E_Q(|S_\tau^i|) < \infty$ and $E_Q(S_\tau^i) = E_Q(S_0^i)$ for each i by the optional stopping theorem. But then S is a uniformly integrable $(\mathbb{F}, Q)$ martingale by Lemma 8, and we may conclude by Theorem 1. ■

With respect to the model described in Section 2 and the information sets discussed in the finance literature, efficiency of $(\mathbb{F}, S)$ is called semi-strong efficiency, since in our economy $\mathbb{F}$ corresponds to publicly available information. Theorem 5 then proves that semi-strong form efficiency implies weak-form efficiency. Weak-form efficiency corresponds to the information set generated by past security prices $(\mathbb{F}^S, S)$, and in our economy $\mathbb{F}^S \subset \mathbb{F}$. In contrast, strong-form efficiency, inside information, corresponds to an information set expansion. This is discussed in the next section.

4.2 Filtration Expansion

For market efficiency under information expansion, we start with an efficient market $(\mathbb{F}, S)$ and consider a larger filtration $\mathbb{G} \supset \mathbb{F}$. In general, it is well known in the finance literature (e.g. Fama [22], p. 388, Jensen [37], p. 97) that when the information set is expanded to include inside information, market efficiency need not be preserved. Using our characterization theorems, we can easily confirm these insights with a simple example. In this example, the additional information is knowing the risky security's price at a later date. Given this information, an arbitrage strategy is easily constructed.

Consider a market consisting of only two assets, the money market account and a single risky security. Let the risky security's price process be $S_t^1 = \exp(B_t - \frac{1}{2}t)$ where B is a Brownian motion on $[0, 1]$ with the natural filtration $\mathbb{F}$. We know that the market $(\mathbb{F}, S)$ is efficient since there exists a martingale probability measure. Indeed, S is already a martingale under P.

Next, consider the inside information set $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le 1}$ where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(S_1^1)$ represents all information, including the future realizations of the risky security's time 1 value. This information is known at time 0. Then, one can show (see Itô [30]) that S^1 is a $\mathbb{G}$ semimartingale. The market $(\mathbb{G}, S)$ is not efficient. Indeed, consider the admissible strategy $H_t = \mathbf{1}_{\{S_1^1 \ge 2\}} \mathbf{1}_{(0,1]}(t)$ whose final payoff is $(S_1^1 - 1)\mathbf{1}_{\{S_1^1 \ge 2\}}$. If $P\{S_1^1 \ge 2\} > 0$, then this admissible strategy is an arbitrage opportunity. Hence, NA is violated, thus also ND and NFLVR. Therefore, by Theorem 1, the market based on the augmented information set $(\mathbb{G}, S)$ is not efficient.

A different and perhaps more important question in this context is the following: if $(\mathbb{F}, S)$ is efficient and $(\mathbb{G}, S)$ satisfies NFLVR, when is $(\mathbb{G}, S)$ efficient? We know, via Theorem 1, that a necessary and sufficient condition is that ND holds also for $(\mathbb{G}, S)$. The next section gives an explicit example where passing from $(\mathbb{F}, S)$ to $(\mathbb{G}, S)$ can yield an inefficient market, which however still satisfies NFLVR.

4.2.1 Example (An NFLVR but Inefficient Market)

We now give an example of a market $(\mathbb{F}, S)$ that is efficient, and where under information expansion $\mathbb{G} \supset \mathbb{F}$, the market $(\mathbb{G}, S)$ satisfies NFLVR but not ND. The example is based on a construction by Delbaen and Schachermayer [18], which we repeat here for clarity of the presentation. The time set is $[0, \infty]$ and the values at infinity of all involved processes are determined by their limits as $t \to \infty$, which always exist.

Let the filtration $\mathbb{F}$ be the natural filtration generated by two independent Brownian motions W and B. In this example we take $\mathcal{F} = \mathcal{F}_\infty$. Define the stopping times

$$\tau = \inf\{t \ge 0 : \mathcal{E}(W)_t = 2\} \qquad \text{and} \qquad \rho = \inf\{t \ge 0 : \mathcal{E}(B)_t = 1/2\}$$

where $\mathcal{E}(B)_t = \exp(B_t - \frac{1}{2}t)$ is the stochastic exponential of the Brownian motion B, and similarly for $\mathcal{E}(W)$. Define processes S and Z by

$$S = \mathcal{E}(B)^{\tau\wedge\rho} \qquad \text{and} \qquad Z = \mathcal{E}(W)^{\tau\wedge\rho}.$$

Lemma 9 (Delbaen and Schachermayer [18]) *The following statements hold:*

(i) S is a non-uniformly integrable P local martingale.

(ii) Z is a uniformly integrable P- martingale with $Z_\infty > 0$ a.s. and $EZ_\infty = 1$.

(iii) SZ is a uniformly integrable P- martingale, implying that S is a uniformly integrable martingale under the measure $Q \sim P$ given by $dQ = Z_\infty dP$.

(iv) $P(\tau < \infty) = \frac{1}{2}$.

The next step is to construct a filtration $\mathbb{G} \supset \mathbb{F}$ such that the price process S still satisfies NFLVR ($\mathcal{M}_{loc}(\mathbb{G}) \neq \emptyset$), but no $R \in \mathcal{M}_{loc}(\mathbb{G})$ exists under which S becomes uniformly integrable. We let $\mathbb{G}$ be the *initial expansion* of $\mathbb{F}$ with the stopping time τ, i.e. the right-continuous completion of

$$\mathbb{F} \vee \sigma(\tau) = (\mathcal{F}_t \vee \sigma(\tau))_{t\geq 0}.$$

(Note that $\mathcal{G}_\infty = \mathcal{F}_\infty = \mathcal{F}$.) Initial expansions of filtrations have been studied extensively by several authors, see e.g. Jacod [32] and the book [38]. However, our example is sufficiently simple that we do not need the general theory of initial expansions.

Lemma 10 *The process B is Brownian motion with respect to* $(\mathbb{G}, P)$*.*

Proof. Fix $0 \leq s < t < \infty$. The distribution under P of $B_t - B_s$ does not depend on the filtration, so it remains normally distributed with zero mean and variance $t - s$. Moreover, B is certainly $\mathbb{G}$ adapted. It remains to prove that $B_t - B_s$ is independent of $\mathcal{G}_s$ under P. Note that the filtration $\mathbb{G}$ is the right-continuous completion of

$$(\mathcal{G}^0_t)_{t\geq 0} = (\mathcal{F}^B_t \vee \mathcal{F}^W_t \vee \sigma(\tau))_{t\geq 0},$$

where $(\mathcal{F}^B_t)_{t\geq 0}$ and $(\mathcal{F}^W_t)_{t\geq 0}$ denote the natural filtrations of B and W, respectively. Pick any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, and define $F = f(B_t - B_s)$. Let X, Y, and Z be bounded random variables measurable with respect to $\mathcal{F}^B_s$, $\mathcal{F}^W_s$, and $\sigma(\tau)$, respectively. Since FX is $\mathcal{F}^B_\infty$-measurable, YZ is $\mathcal{F}^W_\infty$-measurable, and B and W are independent under P, it follows that FX and YZ are independent under P. Similarly, X and YZ are independent. Moreover, since B is Brownian motion, F is independent of $\mathcal{F}^B_s$, and thus of X. This yields

$$E_P(FXYZ) = E_P(FX)E_P(YZ) = E_P(F)E_P(X)E_P(YZ) = E_P(F)E_P(XYZ).$$

By the Monotone Class Theorem, we get $E_P(Fg) = E_P(F)E_P(g)$ for every bounded, $\mathcal{G}^0_s$-measurable g. Now let $F^\varepsilon = f(B_t - B_{s+\varepsilon})$ for $\varepsilon > 0$ small, and pick any bounded, $\mathcal{G}_s$-measurable g. Then g is $\mathcal{G}^0_{s+\varepsilon}$-measurable, so by the above, $E_P(F^\varepsilon g) = E_P(F^\varepsilon)E_P(g)$. Letting $\varepsilon \downarrow 0$ and using continuity and boundedness of f, we obtain $E_P(Fg) = E_P(F)E_P(g)$. This suffices to conclude that $B_t - B_s$ and $\mathcal{G}_s$ are independent. ■

As a consequence of Lemma 10 and the fact that $\tau\wedge\rho$ is a $\mathbb{G}$ stopping time, $S = \mathcal{E}(B)^{\tau\wedge\rho}$ remains a $(\mathbb{G}, P)$ local martingale. In particular, S satisfies NFLVR with respect to $\mathbb{G}$. However, the following result shows that ND fails, which completes our example.

Theorem 6 *The market* $(\mathbb{G}, S)$ *constructed above does not satisfy ND.*

Proof. We will prove that $\mathcal{M}(\mathbb{G}) = \emptyset$. Define the $\mathbb{G}$ adapted process $\tilde{S} = \mathbf{1}_{\{\tau=\infty\}}S$. We claim that if S is a $(\mathbb{G}, R)$ uniformly integrable martingale for some $R \sim P$, then so is $\tilde{S}$. Indeed, in this case

$$\tilde{S}_t = \mathbf{1}_{\{\tau=\infty\}}S_t = \mathbf{1}_{\{\tau=\infty\}}E_R(S_\infty \mid \mathcal{G}_t) = E_R(\mathbf{1}_{\{\tau=\infty\}}S_\infty \mid \mathcal{G}_t),$$

so that $\tilde{S}$ is closed by $\mathbf{1}_{\{\tau=\infty\}}S_\infty$. Suppose for contradiction that such an R exists. Then

$$E_R(\tilde{S}_\infty) = E_R(\tilde{S}_0) = R(\tau = \infty).$$

On the other hand,

$$E_R(\tilde{S}_\infty) = E_R(\mathbf{1}_{\{\tau=\infty\}}\mathcal{E}(B)_\rho) = \frac{1}{2}R(\tau = \infty).$$

Since $R \sim P$ and $P(\tau = \infty) = \frac{1}{2} > 0$, this is a contradiction. It follows that $\tilde{S}$ cannot be a $(\mathbb{G}, R)$-uniformly integrable martingale for any $R \sim P$, so neither can S. ■

The remainder of this section looks for alternative conditions that imply efficiency (or equivalently ND) under an information set expansion. We discover three sufficient conditions; if the market is either: (i) discrete time, (ii) complete, or (iii) the H-hypothesis holds.

4.2.2 Discrete Time Markets

In a discrete time market, if $(\mathbb{F}, S)$ is efficient and $(\mathbb{G}, S)$ satisfies NFLVR, then $(\mathbb{G}, S)$ is efficient. This follows directly from Theorem 4 since, under this hypothesis, NFLVR is a sufficient condition for the efficiency of $(\mathbb{G}, S)$. For continuous time models, however, the situation is much more complex.

4.2.3 Complete Markets

If $(\mathbb{F}, S)$ is a complete and efficient market and $(\mathbb{G}, S)$ satisfies NFLVR, then $(\mathbb{G}, S)$ is efficient. This follows because in a complete market, strategies which are maximal in the smaller filtration also remain maximal in the larger filtration (subject to certain regularity conditions). Hence, information expansion introduces no new profitable trading strategies. To prove this claim, we start with the definition of a complete market.

We will use the following definition of completeness; it says that there is only one risk-neutral measure on $\mathcal{F}_\infty$.

Definition 6 (Completeness) *A market* $(\mathbb{F}, S)$ *is called* complete *if it satisfies NFLVR and all* $Q \in \mathcal{M}(\mathbb{F})$ *coincide on* $\mathcal{F}_\infty$.

For the rest of this section, *we restrict attention to the case where the security process* S *is strictly positive and* $\mathbb{F}$ *locally bounded*. This guarantees that S is a special semimartingale, which is needed for the proof of the following lemma.

Lemma 11 *Let* S *be an* n*-dimensional, locally bounded* $\mathbb{F}$ *semimartingale with positive components, satisfying NFLVR with respect to* $\mathbb{F}$*. If* $\mathbb{G} \supset \mathbb{F}$ *is a larger filtration, then* $\mathcal{M}_{loc}(\mathbb{G}) \subset \mathcal{M}_{loc}(\mathbb{F})$.

Proof. A theorem by Stricker [54] says that if M is a positive $\mathbb{G}$ local martingale, then it is an $\mathbb{F}$ supermartingale, and if in addition M is $\mathbb{F}$ special, then it is an $\mathbb{F}$ local martingale. Each S^i satisfies these conditions under any $Q \in \mathcal{M}_{loc}(\mathbb{G})$, taking into account that S is locally bounded with respect to $\mathbb{F}$ and hence special. ∎

Theorem 7 *Let* $(\mathbb{F}, S)$ *be a complete market, and suppose that* S *is strictly positive and locally bounded. If* $\mathbb{G} \supset \mathbb{F}$ *is a larger filtration such that* $(\mathbb{G}, S)$ *satisfies NFLVR, then every locally bounded* $\mathbb{F}$*-maximal strategy is* $\mathbb{G}$*-maximal.*

In particular, if $(\mathbb{F}, S)$ *is efficient, then so is* $(\mathbb{G}, S)$.

Proof. Since S satisfies NFLVR with respect to $\mathbb{G}$, it is a $\mathbb{G}$ semimartingale. By Theorem IV.33 in [49], the stochastic integral $H \cdot S$ does not depend on the filtration ($\mathbb{F}$ or $\mathbb{G}$) as long as H is $\mathbb{F}$ predictable and locally bounded. Now, let H be a locally bounded, $\mathbb{F}$-maximal strategy. Then $E_Q(H \cdot S)_\infty = 0$ for some $Q \in \mathcal{M}_{loc}(\mathbb{F})$ by Lemma 1. However, $(\mathbb{G}, S)$ satisfies NFLVR, so with Lemma 11 and the completeness assumption we get that

$$\emptyset \neq \mathcal{M}_{loc}(\mathbb{G}) \subset \mathcal{M}_{loc}(\mathbb{F}) = \{Q\}.$$

Therefore $Q \in \mathcal{M}_{loc}(\mathbb{G})$, so another application of Lemma 1 shows that H is $\mathbb{G}$-maximal. Finally, ND and hence completeness of $(\mathbb{G}, S)$ now follows from the fact that the strategies $H^i = (0, \ldots, 0, 1, 0, \ldots, 0)$, which are $\mathbb{F}$-maximal by assumption, are also $\mathbb{G}$-maximal. ∎

An interpretation of Theorem 7 is that given a complete and efficient market $(\mathbb{F}, S)$, any additional information that introduces inefficiencies in $(\mathbb{G}, S)$ will in fact introduce arbitrage opportunities as well, in the sense of NFLVR.

4.2.4 Hypothesis H

This section shows that if $(\mathbb{F}, S)$ is an efficient market, $(\mathbb{G}, S)$ satisfies NFLVR, and $\mathbb{G} \supset \mathbb{F}$ is such that the Hypothesis H holds, then $(\mathbb{G}, S)$ is efficient. Hypothesis H refers to the property that given two nested filtrations $\mathbb{F} \subset \mathbb{G}$ and a probability P, any $(\mathbb{F}, P)$ martingale is again a $(\mathbb{G}, P)$ martingale. An alternative terminology is that $\mathbb{F}$ *is immersed in* $\mathbb{G}$ *under* P.

In modeling credit risk, information expansion and reduction are important considerations. First, differential information characterizes the relationship between structural and reduced form credit risk models. A reduced form model can be obtained via information reduction in a structural model (see Jarrow and Protter [34] for a review). Second, within a reduced form credit risk model, an economy is often characterized by the evolution of a set of state variables yielding the information set $\mathbb{F}$. And, default information is usually included via an expansion of this filtration to include the information generated by a set of default times, yielding the larger information set $\mathbb{G}$. One then studies the conditions under which the martingale pricing technology extends from $\mathbb{F}$ to $\mathbb{G}$. The H-hypothesis guarantees this martingale pricing extension, see Elliott, Jeanblanc and Yor [21] and Bielecki and Rutkowski [3]. It is not surprising, therefore, that the H-hypothesis also plays an important role in understanding information expansion with respect to market efficiency.

The following characterization of Hypothesis H is due to Brémaud and Yor [4].

Theorem 8 (Brémaud-Yor) *The following are equivalent:*

(i) Hypothesis H holds between $\mathbb{F}$ and $\mathbb{G}$ under the measure P.

(ii) $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$. That is, for every $\mathcal{F}_\infty$-measurable nonnegative F and $\mathcal{G}_t$-measurable nonnegative G_t,

$$E_P(FG_t \mid \mathcal{F}_t) = E_P(F \mid \mathcal{F}_t)E_P(G_t \mid \mathcal{F}_t).$$

The next result was proved by Coculescu, Jeanblanc and Nikeghbali [9] in the special case of progressive expansions with random times. Our argument uses the same idea, but the expanded filtration $\mathbb{G} \supset \mathbb{F}$ is now completely general.

Lemma 12 *Suppose that $Q \in \mathcal{M}_{loc}(\mathbb{F})$ and that Hypothesis H holds between $\mathbb{F}$ and $\mathbb{G}$ under some equivalent measure $R \sim Q$. Then there is $Q^* \in \mathcal{M}_{loc}(\mathbb{F})$ such that $\mathbb{F}$ is immersed in $\mathbb{G}$ under Q^*, and $Q^* = Q$ on $\mathcal{F}_\infty$.*

Proof. Let $Z = E_R(\frac{dQ}{dR} \mid \mathcal{F}_\infty)$ and define Q^* via $dQ^* = ZdR$. Then for $A \in \mathcal{F}_\infty$,

$$E_{Q^*}(\mathbf{1}_A) = E_R(Z\mathbf{1}_A) = E_R\left(E_R\left(\frac{dQ}{dR}\mathbf{1}_A \mid \mathcal{F}_\infty\right)\right) = E_Q(\mathbf{1}_A),$$

so $Q = Q^*$ on $\mathcal{F}_\infty$. In particular, then, $Q^* \in \mathcal{M}_{loc}(\mathbb{F})$. Now, choose any $\mathcal{F}_\infty$-measurable $F \geq 0$ and $\mathcal{G}_t$-measurable $G_t \geq 0$. Bayes' rule, immersion under R, and the fact that Z is $\mathcal{F}_\infty$-measurable and nonnegative yields

$$E_{Q^*}(FG_t \mid \mathcal{F}_t) = \frac{E_R(ZFG_t \mid \mathcal{F}_t)}{E_R(Z \mid \mathcal{F}_t)} = \frac{E_R(ZF \mid \mathcal{F}_t)}{E_R(Z \mid \mathcal{F}_t)}E_R(G_t \mid \mathcal{F}_t) = E_{Q^*}(F \mid \mathcal{F}_t)E_R(G_t \mid \mathcal{F}_t).$$

Similarly we obtain

$$E_{Q^*}(G_t \mid \mathcal{F}_t) = \frac{E_R(ZG_t \mid \mathcal{F}_t)}{E_R(Z \mid \mathcal{F}_t)} = E_R(G_t \mid \mathcal{F}_t).$$

Hence $E_{Q^*}(FG_t \mid \mathcal{F}_t) = E_{Q^*}(F \mid \mathcal{F}_t)E_{Q^*}(G_t \mid \mathcal{F}_t)$, so immersion holds under Q^*, as desired. ■

We now give the key theorem of this section. We note that Hypothesis H only has to hold under some arbitrary equivalent measure, not necessarily P or some $Q \in \mathcal{M}_{loc}(\mathbb{F})$.

Theorem 9 *Let $(\mathbb{F}, S)$ be a market that satisfies NFLVR. Suppose that $\mathbb{G} \supset \mathbb{F}$ is a larger filtration such that Hypothesis H holds between $\mathbb{F}$ and $\mathbb{G}$ under some equivalent measure. Then $(\mathbb{G}, S)$ satisfies NFLVR, and every locally bounded $\mathbb{F}$-maximal strategy is $\mathbb{G}$-maximal.*

In particular, if $(\mathbb{F}, S)$ is efficient, then so is $(\mathbb{G}, S)$.

Proof. By Lemma 12, the intersection $\mathcal{M}_{loc}(\mathbb{F}) \cap \mathcal{M}_{loc}(\mathbb{G})$ is non-empty, so $(\mathbb{G}, S)$ satisfies NFLVR. Let H be locally bounded and $\mathbb{F}$-maximal, so that $E_Q(H \cdot S)_T = 0$ for some $Q \in \mathcal{M}_{loc}(\mathbb{F})$. By Lemma 12 there is $Q^* \in \mathcal{M}_{loc}(\mathbb{G})$ coinciding with Q on $\mathcal{F}_T$, so $E_{Q^*}(H \cdot S)_T = 0$ and H is $\mathbb{G}$-maximal. As in the proof of Theorem 7, the local boundedness of H ensures that $H \cdot S$ does not depend on the filtration. Also as in the proof of Theorem 7, the efficiency of $(\mathbb{G}, S)$ follows from the fact that the strategies $H^i = (0, \dots, 0, 1, 0, \dots, 0)$ remain maximal in $\mathbb{G}$. ■

5 Market Efficient Price Processes

In this section we consider some models for price processes useful for pricing options on equities and equity indices. We investigate when these price processes are consistent with market efficiency.

The time set will always be $[0, T]$ for some $T < \infty$. We first consider quite general local volatility models, where a certain dichotomy is present: if NFLVR holds, then either $\mathcal{M}_{loc} = \mathcal{M}$ or $\mathcal{M} = \emptyset$. In the first case, by Theorem 1, the market $(\mathbb{F}, S)$ is efficient, while in the second case it is not. We also look at a class of stochastic volatility models and give sufficient conditions for efficiency. Our goal is to show that there are large classes of efficient models, many of them with price processes that are strict local martingales with respect to the measure under which their dynamics would typically be specified. Results in this vein are well known in the one-dimensional case. In contrast, our results are established in the multi-dimensional case, which is the appropriate setting since $(\mathbb{F}, S)$ should be thought of as a model for an entire market.

These results have two uses. First, they provide an alternative method for testing market efficiency based on a joint hypothesis. Here the joint hypothesis is the specification of a particular stochastic process for asset prices. This additional hypothesis is testable independently of market efficiency. And, an efficient market is a nested subset—the price process supports efficiency if its parameters are in a particular subset and it is inefficient otherwise. In contrast, the classical joint hypothesis—specifying a particular equilibrium model—is not independently testable. The equilibrium model and efficiency are both accepted or rejected in unison.

Second, these results are useful for pricing securities in positive net supply when one wants to impose more structure on the price process than just NFLVR. In particular, one may only want to consider price processes that are consistent with some economic equilibrium, or alternatively stated, are consistent with an efficient market. Our characterization theorems enable one to understand the additional structure required. Such restrictions have already proven useful in the context of asset price bubbles, see Jarrow, Protter and Shimbo [35], [36].

5.1 Local Volatility Models

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let W be d-dimensional Brownian motion with its natural filtration $\mathbb{F}$. We work on the time interval $[0, T]$. Assume that the price process $S = (S^1, \ldots, S^n)$ is governed by the following system of stochastic differential equations.

$$dS_t^i = \sigma^i(S_t, t)dW_t + b^i(S_t, t)dt \qquad (i = 1, \ldots, n), \tag{3}$$

where $\sigma^i : \mathbb{R}^n \times [0, T] \to \mathbb{R}^d$ and $b^i : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ are such that a strong solution exists with $S_t^i > 0$ for all $t \in [0, T]$.

Assume now that NFLVR holds, so that $\mathcal{M}_{loc}(\mathbb{F}) \neq \emptyset$. By the martingale representation theorem, the density process $Z_t = E_P(\frac{dQ}{dP} \mid \mathcal{F}_t)$ associated with some $Q \in \mathcal{M}_{loc}(\mathbb{F})$ can be expressed as $dZ_t = Z_t \theta_t dW_t$ for some adapted, $\mathbb{R}^d$-valued process θ that depends on Q. Defining $W^Q = W - \int_0^{\cdot} \theta_s ds$, Girsanov's theorem implies that

$$dS_t^i = \sigma^i(S_t, t)dW_t^Q + (\sigma^i(S_t, t)\theta_t + b^i(S_t, t))dt \qquad (i = 1, \ldots, n).$$

Since S^i is a local martingale under Q, the drift term is identically zero, so that

$$dS_t^i = \sigma^i(S_t, t)dW_t^Q \qquad (i = 1, \ldots, n).$$

Now, W^Q is Brownian motion under Q, so we deduce that S has the same law under every $Q \in \mathcal{M}_{loc}(\mathbb{F})$. This immediately yields the following theorem, which, although well-known, we state due to its relevance in the present context.

Theorem 10 *If the local volatility model described in (3) satisfies NFLVR, then it is either a true martingale under every $Q \in \mathcal{M}_{loc}$ and $(\mathbb{F}, S)$ is efficient, or it is a strict local martingale under every $Q \in \mathcal{M}_{loc}$ and $(\mathbb{F}, S)$ is inefficient.*

Which of the two possibilities actually holds is determined entirely by the properties of σ. Necessary and sufficient conditions under various regularity assumptions on σ have been investigated by several authors, see for example Carr, Cherny and Urusov [6], Cheridito, Filipovic and Yor [7], and Mijatovic and Urusov [47]. For example, in the case where $n = 1$ and $\sigma^1(x, t) = \sigma(x)$ for some measurable function $\sigma(\cdot)$ satisfying weak regularity conditions, the price process is a true martingale under Q if and only if for some $c > 0$,

$$\int_c^\infty \frac{x}{\sigma(x)^2} dx = \infty,$$

see Carr, Cherny and Urusov [6] for details.

We remark that the question of whether the local volatility model described above satisfies NFLVR or not is less interesting; this is almost always assumed, and the risk-neutral dynamics are then specified directly (i.e. one does not model the b^i.)

5.2 Stochastic Volatility Models

We consider a class of stochastic volatility models where the correlation structure between the different processes does not change with time. We expand upon earlier work of Sin [53], who considers a similar model in the one-dimensional case. See also Hobson [28], who investigates related problems in the one-dimensional case.

We work on $[0,T]$, with W being d-dimensional Brownian motion on $(\Omega,\mathcal{F},P)$ and $\mathbb{F}$ its natural filtration. The model is given by the following system of stochastic differential equations.

$$\begin{aligned} dS_t^i &= S_t^i f^i(v_t,t)\sigma_i dW_t && (i=1,\dots,n) \\ dv_t^j &= a_j dW_t + b^j(v_t^j,t)dt && (j=1,\dots,m). \end{aligned}$$

Here $\sigma_i, a_j \in \mathbb{R}^d$ for $i=1,\dots,n$ and $j=1,\dots,m$. Moreover, each $b^j:\mathbb{R}\times[0,T]\to\mathbb{R}$ is assumed to be Lipschitz. This guarantees that the SDE for $v_t=(v_t^1,\dots,v_t^m)$ has a strong (non-explosive) solution on $[0,T]$. If, for instance, $f^i:\mathbb{R}^m\times[0,T]\to\mathbb{R}_+$ is locally bounded for each i, the local martingales

$$S_t^i = S_0^i \exp\Big(\int_0^t f^i(v_s)\sigma_i dW_s - \frac{1}{2}|\sigma_i|^2\int_0^t f^i(v_s)^2 ds\Big), \qquad i=1,\dots,n,$$

stay strictly positive (we assume that $S_0^i>0$ for all i.) This will be the case under the conditions we will impose on the f^i. Notice that NFLVR is automatically satisfied since each S^i is a local martingale under the original measure. Specifying the model in this way is typical in applications, and allows us to focus on the question of whether ND holds.

We will impose the following condition on the model.

Condition 1 *The functions f^i are Lipschitz on $(-\infty,C]^m$ for every $C>0$. More precisely, there exist constants K_C such that for $i=1,\dots,n$,*

$$|f^i(y,t)-f^i(z,t)|\le K_C|y-z|$$

for every $y,z\in\mathbb{R}^m$ with $y^j\le C$, $z^j\le C$, $j=1,\dots,m$.

At first, this condition may seem restrictive and somewhat arbitrary. However, given that $f^i(y,t)$ is always nonnegative and should be thought of as being increasing in each volatility component y^j, the condition makes more sense. Notice that it only imposes very mild restrictions on the growth rate of $f^i(y,t)$ as the components of y become large.

An important special case where the Lipschitz condition on b^j holds is when $b^j(v_t,t)=\rho_j(\kappa_j-v_t^j)$ for some positive constants ρ_j and κ_j, i.e. where the volatilities are mean-reverting. This is similar to the situation considered by Sin [53].

We now state the main theorem of this section. It provides sufficient conditions guaranteeing that ND holds. In what follows, 'prime' denotes transpose.

Theorem 11 *Consider the stochastic volatility model with constant correlations described above, and assume that Condition 1 is satisfied. If there is a vector $\theta\in\mathbb{R}^d$ such that for all i and j,*

$$\theta'\sigma_i=0, \qquad \theta' a_j\ge\sigma_i' a_j, \qquad \theta' a_j\ge 0,$$

then $\mathcal{M}\neq\emptyset$. If $\sigma_i' a_j\le 0$ for all i and all j, then S is already a martingale under P.

The following corollary gives a simple geometric condition that guarantees the existence of the vector θ required in Theorem 11. For a set of vectors $y_1, \ldots, y_n$, let $\mathrm{conv}(y_1, \ldots, y_n)$ denote their convex hull, and $\mathrm{span}(y_1, \ldots, y_n)$ their linear span.

Corollary 2 *Consider the stochastic volatility model with constant correlations described above, and assume that Condition 1 is satisfied. If*

$$conv(a_1, \ldots, a_m) \cap span(\sigma_1, \ldots, \sigma_n) = \emptyset,$$

then $\mathcal{M} \neq \emptyset$.

Proof. Since $\mathrm{conv}(a_1, \ldots, a_m)$ is compact and convex, and $\mathrm{span}(\sigma_1, \ldots, \sigma_n)$ is closed and convex they can be strictly separated by a hyperplane. In particular, there exists $\theta \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$ such that $\theta' a_j > \alpha$ for all j and $\theta'(\lambda \sigma_i) \leq \alpha$ for all i and all $\lambda \in \mathbb{R}$. Take $\lambda = \pm 1$ to see that $\alpha = 0$ and $\theta' \sigma_i = 0$ for all i. By positive scaling we may assume that $\theta' a_j \geq \sigma_i' a_j$ for all i and j. Apply Theorem 11 with this θ. ■

The proof of Theorem 11 requires two lemmas, both of which are similar to results that are well-known in the literature. The first lemma is a slight modification of a comparison theorem due to Ikeda and Watanabe, see [29], Theorem 1.1.

Lemma 13 *Suppose that for* $j = 1, 2$ *and some continuous* $a : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^d$, *we have*

$$Y_t^j = Y_0^j + \int_0^t a(Y_s^j, s) dW_s + \int_0^t \beta_s^j ds,$$

where W *is* d*-dimensional Brownian motion and* β^j *are adapted processes. Suppose the following conditions are satisfied:*

(i) $\beta_t^1 \geq b^1(Y_t^1, t)$ *and* $b^2(Y_t^2, t) \geq \beta^2(t)$ *for some measurable functions* b^1, b^2 *with* $b^1(y, t) \geq b^2(y, t)$ *for all* y *and* t.

(ii) There is an increasing $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\rho(0) = 0$, $\int_{0+} \rho(u)^{-2} du = \infty$ *such that for all* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$, a *satisfies*

$$|a(x, t) - a(y, t)| \leq \rho(|x - y|)$$

(iii) $Y_0^1 \geq Y_0^2$.

(iv) Pathwise uniqueness holds for one of $dY_t = a(Y_t, t) dW_t + b^j(Y_., t) dt$, $j = 1, 2$.

Then $Y_t^1 \geq Y_t^2$ *for all* t.

Proof. Theorem 1.1 in [29] contains the above statement, but for the case $d = 1$. However, the proof remains valid for our setup. ■

The second lemma uses the same well-known techniques as the proof of Lemma 4.2 in Sin [53]. See also [6], [7], [47]. For completeness and since the proof is quite short, we provide the details in the appendix. Thanks are due to Younes Kchia, who pointed out an error in an earlier version of this lemma.

Lemma 14 *Let Y be an n-dimensional diffusion on $[0,T]$ satisfying a stochastic differential equation*

$$dY_t = A(Y_t,t)dW_t + b(Y_t,t)dt,$$

where W is d-dimensional Brownian motion and A and b are measurable functions with values in $\mathbb{R}^{n\times d}$ and $\mathbb{R}^n$, respectively. Assume that a non-explosive solution exists and is pathwise unique on $[0,T]$. If f is an $\mathbb{R}^d$-valued locally Lipschitz function such that the auxiliary SDE

$$d\hat{Y}_t = A(\hat{Y}_\cdot,t)dW_t + [b(\hat{Y}_\cdot,t) + A(\hat{Y}_\cdot,t)f(\hat{Y}_\cdot,t)]dt, \qquad \hat{Y}_0 = Y_0 \tag{4}$$

has a non-explosive and pathwise unique solution on $[0,T]$, then the positive local martingale X given by

$$X_t = \exp\Big(\int_0^t f(Y_s,s)dW_s - \frac{1}{2}\int_0^t |f(Y_s,s)|^2 ds\Big)$$

is a true martingale on $[0,T]$.

Proof of Theorem 11. The goal is to find a measure $Q \sim P$ under which each S^i becomes a martingale. The proof proceeds in a number of steps.

Step 1. As a candidate density process for a measure change, let Z be the stochastic exponential of $-\int_0^\cdot h(v_t,t)\theta' dW_t$, where we define $h : \mathbb{R}^m \times [0,T] \to \mathbb{R}$ by $h(y,t) = \max_{i=1,\dots,n} f^i(y,t)$. Then Z is the unique solution of

$$dZ_t = -Z_t h(v_t,t)\theta dW_t, \qquad Z_0 = 1. \tag{5}$$

Since v_t is non-explosive, Z is a strictly positive local martingale. Lemma 14 implies that it is a true martingale if $\hat{v}_t$ is non-explosive and pathwise unique, where

$$d\hat{v}^j_t = a_j dW_t + \Big[b^j(\hat{v}^j_t,t) - h(\hat{v}_t,t)a_j'\theta\Big]dt, \quad \hat{v}^j_0 = v^j_0 \qquad (j=1,\dots,m).$$

Step 2. Due to Condition 1, $\hat{v}_t$ is non-explosive and pathwise unique at least up to τ_k, where

$$\tau_k = \inf\{t \ge 0 : \max_{j=1,\dots,m} \hat{v}^j_t \ge k\}.$$

We need to show that, almost surely, $\tau_k \ge T$ for large enough k. Since $a_j'\theta \ge 0$, the drift coefficient of $\hat{v}^j_t$ is bounded above by $b^j(\hat{v}^j_t,t)$. Lemma 13 then shows that $\hat{v}^j_t \le w^j_t$ up to time τ_k, where w^j is the solution of

$$dw^j_t = a_j dW_t + b^j(w^j_t,t)dt, \quad w_0 = v^j_0,$$

which is pathwise unique. Note that the condition on the volatility coefficient in Lemma 13 is satisfied since a_j is constant. Since b^j is Lipschitz, each w^j is non-explosive and we deduce that no $\hat{v}^j$ can explode to $+\infty$. This shows that $\tau_k \ge T$ for large enough k.

Step 3. From Steps 1–2 it follows that Z is a true martingale on $[0,T]$, so it is the density process of the measure Q given by $dQ = Z_T dP$. Then $dB_t = dW_t + h(v_t,t)\theta dt$ is

Brownian motion under Q by Girsanov's theorem, and the dynamics of S and v can be written

$$\begin{aligned}
dS_t^i &= S_t^i f^i(v_t,t)\sigma_i dB_t && (i=1,\dots,n)\\
dv_t^j &= a_j dB_t + \Big[b^j(v_t^j,t) - h(v_t,t)a_j'\theta\Big]\,dt && (j=1,\dots,m),
\end{aligned}$$

taking into account that $\theta'\sigma_i = 0$ for all i. The auxiliary SDE associated with S^i is

$$d\hat{v}_t^j = a_j dB_t + \Big[b^j(\hat{v}_t^j,t) + f^i(\hat{v}_t,t)\sigma_i' a_j - h(\hat{v}_t,t)\theta' a_j\Big]\,dt, \quad \hat{v}_0^j = v_0^j \qquad (j=1,\dots,m).$$

Since $\theta' a_j \geq \sigma_i' a_j$ and $h(\hat{v}_t,t) \geq f^i(\hat{v}_t,t)$, the drift coefficient is bounded above by $b^j(\hat{v}_t^j,t) + f^i(\hat{v}_t,t)[\sigma_i' a_j - \theta' a_j] \leq b^j(\hat{v}_t^j,t)$. The same argument as in Step 2 shows that $\hat{v}_t$ does not explode on $[0,T]$. This proves that S^i is a martingale under Q for each i and finishes the proof of part (i) of the theorem.

To prove the last assertion, notice that if $\langle \sigma^i, a^j\rangle \leq 0$ for all i and j, then $\theta = 0$ works. Therefore S is already a martingale under the original measure. ■

The larger $d-m$, the "easier" it is for condition (i) in Theorem 11 to be satisfied. In particular, it holds if $m=1$ and a^1 is not in the span of $\sigma^1,\dots,\sigma^n$. On the other hand, if $\sigma^1,\dots,\sigma^n$ span all of $\mathbb{R}^d$, then of course condition (i) always fails. This is the case of a complete market. It should however be emphasized that Theorem 11 only gives *sufficient* conditions for checking (ND).

One noteworthy special case where part (ii) of Theorem 11 applies is when each of the vectors a^j is orthogonal to all the σ^i. In this case there are, after a change of coordinates, two independent sets of Brownian motions, one of them driving the S^i and the other driving the v^j.

In general we cannot expect the sufficient conditions of Theorem 11 to also be necessary for ND. This is because they are independent of the choice of f^i and b^j. By choosing appropriate f^i, for instance by making them bounded, we can always guarantee that ND holds, independently of $a_1,\dots,a_m$ and $\sigma_1,\dots,\sigma_n$. A weaker result is that under certain conditions on the correlation structure, one can find functions f^i and b^j such that ND fails. (Of course, the f^i we consider should always satisfy the basic assumptions of the model, in particular Condition 1.)

Theorem 12 *Consider the stochastic volatility model with constant correlations, and assume there is a vector $\eta \in conv(a_1,\dots,a_m) \cap span(\sigma_1,\dots,\sigma_n)$ with $\eta'\sigma_k > 0$ for some k. Then there exist functions f^i and b^j that satisfy the model assumptions, such that S^k is a strict local martingale under every $Q \in \mathcal{M}_{loc}$.*

Proof. Assume for notational simplicity that $|\eta| = |\sigma_k| = 1$. Write $\eta = \lambda^1 a_1 + \cdots + \lambda^m a_m$ for convex weights λ^j, and define

$$f^k(y,t) = \exp\Big(\sum_{j=1}^m \lambda^j y^j - \frac{1}{2}t\Big), \qquad f^i(y,t) \equiv 1 \quad (i \neq k),$$

and

$$b^j(y^j,t) \equiv 0 \quad (j = 1,\dots,m).$$

Define also $B^1_t = \eta W_t$ and $B^2_t = \sigma_k W_t$, which are one-dimensional Brownian motions with $d\langle B^1, B^2\rangle_t = \eta'\sigma_k dt$, where $\eta'\sigma_k > 0$. With $u_t = \exp(B_t - \frac{1}{2}t)$, we then have

$$\begin{aligned} dS^k_t &= S^k_t u_t dB^2_t \\ du_t &= u_t dB^1_t. \end{aligned}$$

From Lemma 4.2 and Lemma 4.3 in [53], we deduce that S^k is a strict local martingale. Now, pick an arbitrary $Q \in \mathcal{M}_{loc}$ and let Z be the corresponding density process. By martingale representation, $dZ_t = Z_t\theta_t dW_t$ for some $\mathbb{R}^d$-valued process θ. Since every S^i remains a local martingale under Q, it follows that $\langle Z, S^i\rangle = 0$. But

$$\langle Z, S^i\rangle_t = \int_0^t S^i_s f^i(v_s,s) Z_s \sigma_i' \theta_t dt,$$

so because $S^i_s f^i(v_s,s)Z_s > 0$, we have $\sigma_i'\theta_t = 0$. Since $\eta \in \operatorname{span}(\sigma_1,\dots,\sigma_n)$, we also have $\eta'\theta_t = 0$. Thus B^1 and B^2 are still Brownian motions under Q, so the law of (S^k, u) is unchanged and we deduce that S^k is a strict local martingale under Q. This completes the proof. ■

6 Conclusion

Market efficiency has been a topic discussed and tested in the financial economics literature for over four decades. But, despite this extensive investigation and analysis, there still exist some common misbeliefs regarding the meaning and testing of an efficient market. By formalizing the definition of an efficient market, this paper clarifies these misbeliefs. We prove various theorems relating to an efficient market for understanding empirical testing, profitable trading strategies, and the properties of asset price processes. Perhaps most interesting and in contrast to common belief, we show that one can test an efficient market without specifying a particular equilibrium model. We hope that our mathematical characterizations of market efficiency lead to subsequent research studying its additional implications with respect to both empirical testing and derivatives pricing.

A Appendix

A.1 Proof of Lemma 14

Throughout, P is the measure under which W is Brownian motion. Define stopping times

$$\tau_k = \inf\{t \geq 0 : \int_0^t |f(Y_s,s)|^2 ds \geq k\} \wedge T$$

and processes $X^k = X^{\tau_k} = \exp\{M^k - \frac{1}{2}\langle M^k, M^k\rangle\}$, where $M^k_t = \int_0^{t\wedge\tau_k} f(Y_s,s)dW_s$. By Novikov's criterion, each X^k is a true martingale. It stays strictly positive, so we define equivalent measures Q^k by $dQ^k = X^k_T dP$. By Girsanov's theorem,

$$dY_t = A(Y_t,t)dW^k_t + \left[b(Y_t,t) + \mathbf{1}_{\{t\leq\tau_k\}}A(Y_t,t)f(Y_t,t)\right]dt,$$

where $dW_t^k = dW_t - \mathbf{1}_{\{t\le\tau_k\}} f(Y_t,t)dt$ is Brownian motion under Q^k. Next, define stopping times

$$\hat\tau_k = \inf\{t \geq 0 : \int_0^t |f(\hat Y_s, s)|^2 ds \geq k\} \wedge T.$$

By the non-explosion of Y and $\hat Y$, the stopping times τ_k and $\hat\tau_k$ are equal to T for all sufficiently large k, almost surely. Moreover, by pathwise uniqueness, the law of $\hat\tau_k$ under P is the same as the law of τ_k under Q^k. These facts yield

$$\begin{aligned} E_P(X_T) &= \lim_{k\to\infty} E(X_T \mathbf{1}_{\{\tau_k = T\}}) \\ &= \lim_{k\to\infty} E(X_{T\wedge\tau_k} \mathbf{1}_{\{\tau_k = T\}}) \\ &= \lim_{k\to\infty} Q^k(\tau_k = T) \\ &= \lim_{k\to\infty} P(\hat\tau_k = T) = 1. \end{aligned}$$

This shows that X has constant expectation and hence is a martingale.

A.2 Proof of Lemma 7

Let (T_n) be a localizing sequence for L. For each $t = 1,\dots,T$,

$$\begin{aligned} E(L_t^- \mid \mathcal{F}_{t-1})\mathbf{1}_{\{T_n>t-1\}} &= E(L_t^- \mathbf{1}_{\{T_n>t-1\}} \mid \mathcal{F}_{t-1})\mathbf{1}_{\{T_n>t-1\}} \\ &\geq -L_{t-1}^{T_n}\mathbf{1}_{\{T_n>t-1\}} \\ &= -L_{t-1}\mathbf{1}_{\{T_n>t-1\}}, \end{aligned}$$

where the inequality uses that both sides are zero on $\{T_n < t\}$, whereas on $\{T_n > t-1\}$, $L_t^- \mathbf{1}_{\{T_n>t-1\}} = (L_t^{T_n})^- \geq -L_t^{T_n}$. Let $n\to\infty$ and use $L_t^- \geq 0$ to obtain $E(L_t^- \mid \mathcal{F}_{t-1}) \geq L_{t-1}^-$. This yields $E(L_{t-1}^-) \leq E(L_t^-)$, and since $E(L_T^-) < \infty$, $E(L_t^-) < \infty$ for all $t = 0,\dots,T$. Hence L^- is a submartingale, so for all t and n, $E(L_{t\wedge T_n}^-) \leq E(L_T^-)$.

Next, consider the positive parts L_t^+. By Fatou's lemma,

$$E(L_t^+) \leq \liminf_{n\to\infty} E(L_{t\wedge T_n} + L_{t\wedge T_n}^-) = E(L_0) + \liminf_{n\to\infty} E(L_{t\wedge T_n}^-).$$

By the first part of the proof, this is dominated by $E(|L_0|) + E(L_T^-) < \infty$. We thus obtain $E(|L_t|) < \infty$ for all t.

Finally, for fixed t, $\sup_n |L_t^{T_n}| \leq \max_{t=0,\dots,T} |L_t| \leq \sum_{t=0}^T |L_t|$, which has finite expectation. So by Dominated Convergence,

$$E(L_k \mid \mathcal{F}_{k-1}) = \lim_n E(L_k^{T_n} \mid \mathcal{F}_{k-1}) = \lim_n L_{k-1}^{T_n} = L_{k-1}.$$

This finishes the proof.

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